（1）$\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}\,(0<\theta<\pi)$；

（2）$\sin^2\alpha\sin^2\beta+\cos^2\alpha\cos^2\beta-\frac{1}{2}\cos2\alpha\cos2\beta$；

（3）$\dfrac{(\sin\theta+\cos\theta-1)(\sin\theta-\cos\theta+1)}{\sin2\theta}$.

解：(1) $1+\sin\theta=\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2$，同理，$1-\sin\theta=\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2$

原式 $=\left|\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right|-\left|\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right|=\sin\frac{\theta}{2}+\cos\frac{\theta}{2}-\left|\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right|$

$$=\begin{cases}2\sin\frac{\theta}{2}, & 0<\theta<\frac{\pi}{2}\\ 2\cos\frac{\theta}{2}, & \frac{\pi}{2}\le\theta<\pi\end{cases}$$

(2) $=\frac{1-\cos2\alpha}{2}\cdot\frac{1-\cos2\beta}{2}+\frac{1+\cos2\alpha}{2}\cdot\frac{1+\cos2\beta}{2}-\frac{1}{2}\cos2\alpha\cos2\beta$

$=\frac{1-\cos2\alpha-\cos2\beta+\cos2\alpha\cos2\beta}{4}+\frac{1+\cos2\alpha+\cos2\beta+\cos2\alpha\cos2\beta}{4}-\frac{2\cos2\alpha\cos2\beta}{4}$

$=\frac{2}{4}$

$=\frac{1}{2}$

$$(3) = \frac{\sin^2\theta - (\cos^2\theta - 1)^2}{2\sin\theta\cos\theta} = \frac{\sin^2\theta - \cos^2\theta - 1 + 2\cos\theta}{2\sin\theta\cos\theta} = \frac{-2\cos^2\theta + 2\cos\theta}{2\sin\theta\cos\theta} = \frac{-2\cos\theta + 2}{2\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$\left(= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \tan\frac{\theta}{2} \right)$$

【例 2】证明：$\dfrac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta}=\tan\theta$.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta - \cos^2\theta + \sin^2\theta}{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta}$$

$$= \frac{2\sin^2\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta + \sin\theta\cos\theta}{\cos^2\theta + \sin\theta\cos\theta}$$

$$= \frac{\sin\theta\,(\cos\theta + \sin\theta)}{\cos\theta\,(\sin\theta + \cos\theta)} = \tan\theta$$

【例 3】证明：（1）$3+\cos4\alpha-4\cos2\alpha=8\sin^4\alpha$；（2）$\dfrac{\tan\alpha\tan2\alpha}{\tan2\alpha-\tan\alpha}+\sqrt{3}(\sin^2\alpha-\cos^2\alpha)=2\sin(2\alpha-\dfrac{\pi}{3})$.

$\frac{1-\cos2a}{2}=\sin^2 a$

(1) $3+2\cos^2 2a-1-4\cos2a=8\sin^4\alpha$

$2\cos^2 2\alpha-4\cos2a+2=8\sin^4\alpha$

$2(1-\cos2a)^2=8\sin^4 a$

$2(2\sin^2\alpha)^2=8\sin^4 a$

$$\frac{\frac{\sin\alpha\sin2\alpha}{\cos\alpha\cos2\alpha}}{\frac{\sin2\alpha}{\cos2\alpha}-\frac{\sin\alpha}{\cos\alpha}}+\sqrt{3}\left(\frac{1-\cos2a}{2}-\frac{1+\cos2a}{2}\right)=2\sin(2a-\frac{\pi}{3})$$

$$\frac{\sin\alpha\sin2\alpha}{\sin2\alpha\cos\alpha-\sin\alpha\cos2\alpha}-\sqrt{3}\cos2\alpha=\sin2\alpha-\sqrt{3}\cos2\alpha$$

$$\frac{\sin a\cdot\sin2a}{\sin(2a-a)}-\sqrt{3}\cos2\alpha=\sin2\alpha-\sqrt{3}\cos2a$$

$$\sin2a-\sqrt{3}\cos2a=\sin2a-\sqrt{3}\cos2a$$

【例 4】化简：$\cos^4 20^\circ+\cos^4 40^\circ+\cos^4 80^\circ$.

$$=\frac{(1+\cos40^\circ)^2}{4}+\frac{(1+\cos80^\circ)^2}{4}+\frac{(1+\cos160^\circ)^2}{4}$$

$$=\frac{\cos^2 40^\circ+2\cos40^\circ+1}{4}+\frac{\cos^2 80^\circ+2\cos80^\circ+1}{4}+\frac{\cos^2 160^\circ+2\cos160^\circ+1}{4}$$

$$=\frac{1+\cos80^\circ}{8}+\frac{\cos40^\circ}{2}+\frac{1}{4}+\frac{\cos160^\circ+1}{8}+\frac{\cos80^\circ}{2}+\frac{1}{4}+\frac{\cos320^\circ+1}{8}+\frac{\cos160^\circ}{2}+\frac{1}{4}$$

$$=\frac{3+\cos80^\circ+\cos160^\circ+\cos320^\circ}{8}$$

【例 4】化简：$\cos^4 20^\circ+\cos^4 40^\circ+\cos^4 80^\circ$.

解：$\cos^4\theta=(\cos^2\theta)^2=\left(\dfrac{1+\cos 2\theta}{2}\right)^2=\dfrac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta)$

$$=\frac{1}{4}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)=\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$$

原式 $=\dfrac{3}{8}+\dfrac{1}{2}\cos 40^\circ+\dfrac{1}{8}\cos 80^\circ+\dfrac{3}{8}+\dfrac{1}{2}\cos 80^\circ+\dfrac{1}{8}\cos 160^\circ+\dfrac{3}{8}+\dfrac{1}{2}\cos 160^\circ+\dfrac{1}{8}\cos 320^\circ$

$$=\frac{9}{8}+\frac{1}{2}\cos 40^\circ+\frac{5}{8}\cos 80^\circ+\frac{5}{8}\cos 160^\circ+\frac{1}{8}\cos 320^\circ$$

$$=\frac{9}{8}+\frac{5}{8}\cos 40^\circ+\frac{5}{8}\cos(120^\circ-40^\circ)+\frac{5}{8}\cos(120^\circ+40^\circ)$$

$$=\frac{9}{8}+\frac{5}{8}\cos 40^\circ+\frac{5}{8}\left(-\frac{1}{2}\cos 40^\circ+\cancel{\frac{\sqrt{3}}{2}\sin 40^\circ}-\frac{1}{2}\cos 40^\circ-\cancel{\frac{\sqrt{3}}{2}\sin 40^\circ}\right)$$

$$=\frac{9}{8}+\frac{5}{8}\cos 40^\circ+\frac{5}{8}(-\cos 40^\circ)$$

$$=\frac{9}{8}.$$